B.SC. SIXTH SEMESTER (HONOURS) EXAMINATIONS, 2021

Subject: Mathematics **Course ID: 62116**

Course Code: SH/MTH/603/DSE-3 **Course Title: Number Theory**

Full Marks: 40 **Time: 2 Hours**

<u>The figures in the margin indicate full marks</u>

Notations and symbols have their usual meaning.

1. **Answer *any five* of the following questions:** **(2×5=10)**

a) Show that $\sum_{k=1}^{n} \mu(k!) = 1$.

b) Solve the linear congruence $7x \equiv 3 \ (mod\ 15)$.

c) "If $(m, n) = 1$, then $(\phi(m), \phi(n)) = 1$"—true or false? Justify with proper reason.

d) Find the order of 5 modulo 17.

e) Find the remainder when $333^{333}$is divided by 7.

f) Find the sum of integers less than 100 and prime to 100.

g) If $a$is a primitive root of m, then prove that $a^{\frac{\phi(m)}{2}} \equiv -1 \ (mod\ m)$.

h) Prove that if n is an odd integer, then $n^2 - 1$ is divisible by 8.

2. **Answer *any four* of the following questions:** **(5×4=20)**

a) Find the least positive integer which leaves remainders 2, 3 and 4 when divided by 3, 5 and 11, respectively.

b) (i) If $f(n)$ is a function of n, then prove that $\sum_{d|n} f(d) = \sum_{d|n} f\left(\frac{n}{d}\right)$, where n is a positive integer ≥ 1.

(ii) Prove that $n = \sum_{d|n} \phi(d)$, for an integer $n \geq 1$. 3+2

c) (i) If $f(n)$ is a multiplicative function(not identically zero), then prove that

$$\sum_{d|n} \mu(d) f(d) = (1 - f(p_1))(1 - f(p_2)) \cdots (1 - f(p_k)).$$

(ii) Verify that $1000!$ terminates in 249 zeros. 2+3

d) Prove that:

(i) The necessary and sufficient condition that 'a', when $(a, m) = 1$, to be a primitive root of m is that the numbers $a, a^2, \cdots, a^{\phi(m)}$ forms a reduced residue system modulo m.

(ii) $If\ a$ is a primitive root of p, then prove that $a + p$ is also its primitive root. 2+2+1

e) Prove that if p is an odd prime, then there exists an odd primitive root of p^k $\forall\, k \geq 1$. Also each such primitive root of p^k is a primitive root of $2p^k$.

f) Find the solutions of $3x + 5y + 10z = 151$.

3. Answer *any one* of the following questions: **(10×1=10)**

a) (i) If n is a composite number, then show that $(n) \leq n - \sqrt{n}$.

(ii) If $m = 2^n, n > 2$, then prove that m has no primitive root.

(iii) Prove that $n^5 - n$ is divisible by 5 or 2, for any integer n.

(iv) If $m$and n are relatively prime positive integers, prove that

$m^{\emptyset(n)} + n^{\emptyset(m)} \equiv 1 \ (mod\ mn)$. 2+2+2+4

b) (i) If a is a primitive root of p and p is an odd prime such that a^{p-1} is not congruent to 1 modulo p^2, then show that for every $\alpha \geq 2, a^{\phi(p^{\alpha-1})}$ is not congruent to 1 modulo p^{α}.

(ii) Find the number of zeros at the right end of the integer $141!$.

(iii) Prove Euler's Theorem. 5+2+3

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