B.SC. SIXTH SEMESTER (HONOURS) EXAMINATIONS, 2021

Subject: Mathematics **Course ID: 62112**

Course Code: SH/MTH/602/C-14 **Course Title: Ring Theory and Linear Algebra II**

Full Marks: 40 **Time: 2 Hours**

<u>The figures in the margin indicate full marks</u>

Notations and symbols have their usual meaning.

1. Answer *any five* from the following questions: $2\times 5=10$

a) Find the irreducible elements in a field.

b) Is the polynomial ring $\mathbb{Z}_6[x]$ an integral domain? Justify your answer.

c) Is the polynomial $f(x)=x^3-9x^2+15x-2$ irreducible over $\mathbb{Q}$? Justify your answer.

d) Find the dual space of the vector space $\mathbb{R}^2$ over $\mathbb{R}$.

e) Give an example of a linear operator on $\mathbb{R}^3$ over $\mathbb{R}$ whose characteristic polynomial and minimal polynomial are same.

f) Define a diagonalizable operator on a vector space.

g) Let $(V,<>)$ be an inner product space and S be a subset of V. Prove thatthe orthogonal complement of S is a subspace of V.

h) Let $(V,<>)$ be an inner product space.Prove that if $<\alpha,\beta>=\ 0$ for all $\beta\in V$, then $\alpha=0$.

2. Answer *any four* from the following questions: $5\times 4=20$

a) In a PID, prove that an irreducible element is always prime. Is the ring $\mathbb{Z}[\sqrt{-5}]$ a factorization domain? Justify your answer. $3+2=5$

b) Let R be a commutative ring with identity. Prove that R is a field if and only if $R[x]$ is a PID.

c) (i) Let Wbe an invariant subspace of V under T. Prove that the characteristic polynomial of the restriction operator $T|_W$ divides the characteristic polynomial of T. Also prove that the minimal polynomial of $T|_W$ divides the minimal polynomial of T.

(ii) Give an example of a linear operator T on the vector space $\mathbb{R}^2$ over $\mathbb{R}$ such that the only invariant subspaces under T are $\{0\}$ and $\mathbb{R}^2$. $(2+1)+2=5$

d) Let V be a finite dimensional vector space over the field F and T be a linear operator on V. Define the minimal polynomial of T. If T is diagonalizable, then prove that the minimal polynomial for T is a product of distinct linear factors. $1+4=5$

e) Consider the vector space $\mathbb{R}^4$ over $\mathbb{R}$ with the standard inner product. Let S be the subspace of $\mathbb{R}^4$ consisting of all vectors which are orthogonal to both $(1,0,-1,1)$ and $(2,3,-1,2)$. Find an orthonormal basis for S.

f) Consider the vector space $\mathbb{R}^3$ over $\mathbb{R}$ equipped with the standard inner product. Applying Gram-Schmidt orthogonalization process, orthonormalize the following set of vectors: $\{(3,0,4),(-1,0,7),(2,9,11)\}$.

3. Answer *any one* from the following questions: $\mathbf{10\times 1=10}$

a) (i) Is the ring $\mathbb{Q}[x]\,/<x>$ isomorphic to $\mathbb{Q}$? Justify your answer.

(ii) Consider the polynomial $f(x)=5x^4+4x^3-6x^2-14x+2\in\mathbb{Z}[x]$. Check the irreducibility of f in $\mathbb{Z}$.

(iii) Let F be a field and D denote the differentiation operator on the vector space $F[x]$. Let $W=\{g(x)\in F[x]|\deg(g)\le 20\}$. Prove that $W$is invariant under D.

(iv) Let V be the vector space $\mathbb{C}^2$over $\mathbb{C}$ and let $\mathcal{B}=\{\epsilon_1,\epsilon_2\}$ be the standard ordered basis of V. Let T be the linear operator on V defined by $T(\epsilon_1)=(1+i,2), T(\epsilon_2)=(i,i)$. Using the standard inner product, find T^*. $3+2+2+3=10$

b) (i) Find the gcd of the following polynomials in $\mathbb{Z}_5[x]$: $f(x)=\bar{2}x^5-\bar{2}x^4+\bar{2}x^3-\bar{2}x-\bar{2}$, $g(x)=x^4-\bar{2}x^2+\bar{2}$.

(ii) Find all possible Jordan canonical forms for a linear operator up to similarity whose characteristic polynomial is $(x-2)^4(x-3)^2$ and minimal polynomial is $(x-2)^2(x-3)$.

(iii) Let $\{\alpha_1,\alpha_2,\dots,\alpha_n\}$ be an orthogonal set of non-zero vectors in an inner product space V. If $\beta\in V$ then prove that $\sum_{k=1}^{n}\frac{|<\beta,\alpha_k>|^2}{||\alpha_k||^2}\le ||\beta||^2$. $4+4+2=10$

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