B.SC. SIXTH SEMESTER (HONOURS) EXAMINATIONS, 2021

Subject: Mathematics **Course ID: 62111**

Course Code: SH/MTH/601/C-13 **Course Title: Metric Spaces and Complex Analysis**

Full Marks: 40 **Time: 2 Hours**

The figures in the margin indicate full marks

Notations and symbols have their usual meaning

1. **Answer *any five* of the following questions:** **(2 × 5 = 10)**

a) Let d be the metric define on $\mathbb{N}$, the set of natural numbers, by $d(m,n) = \left|\frac{1}{m} - \frac{1}{n}\right|$, $m, n \in \mathbb{N}$. Prove that $(\mathbb{N}, d)$ is an incomplete metric space.

b) Let A be a subset of a metric space (X, d) and $y \in X$. If there is a sequence $\{x_n\}$ in A converging to y, then show that $y \in \overline{A}$.

c) Are the concepts of compactness and connectedness for subsets of a metric space are dependent? Justify.

d) Prove that the set $A = \{x \in \mathbb{R}: |x| > 0\}$ of $\mathbb{R}$ with usual metric is disconnected.

e) Prove or disprove: "A real function of a complex variable either has derivative zero or the derivative does not exist".

f) Let f be analytic and $|f(z)| < 1$ for $|z| < 1$, prove that $|f^3(z)| \leq \frac{6}{(1-r)^3}$ for $|z| < r < 1$.

g) Show that the function $f(x+iy) = x^3 + ax^2y + bxy^2 + cy^3$, where a, b, c are complex constants, is analytic in $\mathbb{C}$ only if $a = 3i, b = -3, c = -i$.

h) Evaluate $\int_{|z|=1} \frac{z+3}{z^4 + az^3} dz, |a| > 1$.

2. **Answer *any four* of the following questions:** **(5×4= 20)**

a) (i) Let (X, d) be a metric space. Suppose that every real valued continuous function on (X, d) satisfies the intermediate value property. Prove that (X, d) is connected.

(ii) Give an example with justification of a complete metric space which is not compact.

3+2

b) (i) Show that a bounded set A in the set of real numbers $\mathbb{R}$ is totally bounded. Is the converse true? Justify.

(ii) Is a Cauchy sequence in a metric space bounded? Justify. 3+2

c) (i) Let (X, d_1) and (Y, d_2) be metric spaces, and let $f : (X, d_1) \to (X, d_2)$ be a continuous function on (X, d_1) and for each $x \in X$, V is a neighbourhood of $f(x)$ in Y. Is $f^{-1}(V)$ a neighbourhood of x in X. Justify.

(ii) In the metric space $C[0,1]$, the set of all real valued continuous functions on $[0,1]$ with respect to *sup* metric, examine whether $\{f_n\}$ where $f_n(x) = \frac{nx}{n+x}$ is a Cauchy sequence or not.

d) Show that $f(z) = \sqrt{r}e^{i\theta/2}$ $(r > 0,\ -\pi < \theta < \pi)$ is analytic in its domain and $f'(z) = \frac{1}{2f(z)}$. 3+2

e) Let $\gamma(t) = e^{it}, 0 \le t \le 2\pi$. Evaluate $\int_\gamma \frac{\cos z}{z} dz$ and hence deduce that $\int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta) d\theta = 2\pi.$ 2+3

f) (i) Expand $f(z) = \frac{1}{z(z-1)}$ in a Laurent series valid for $1 < |z-2| < 2$.

(ii) If $f(z)$ is differentiable in a region G and $|f(z)|$ is constant in G, then show that $f(z)$ is constant in G. 3+2

3. Answer *any one* of the following questions: (10×1 = 10)

a) (i) Let f be analytic in the domain $D = \{z \in \mathbb{C}: |z| < 2\}$. Prove that

$$2f(0) + f'(0) = \frac{2}{\pi}\int_0^{2\pi} f(e^{i\theta})\cos^2\left(\frac{\theta}{2}\right) d\theta.$$

(ii) A subset Γ of the real line $\mathbb{R}$, with at least two points is connected if Γ is an interval – prove it.

(iii) If every closed ball in a metric space is compact, prove that the metric space is complete. 4+3+3

b) (i) Let (X, d) be a bounded metric space with at least two points. If $T: X \to X$ is a contraction map, then T can not be surjective.

(ii) Prove that $\left|\int_\gamma (z+1)^2 dz\right| \le 9\sqrt{5}$, where $\gamma(t) = 2 - t(2-i), t \in [0,1]$.

(iii) Find the Laurent series for $f(z) = \frac{1}{(z+1)(z-2)^2}$ valid for the annular region given by $0 < |z+1| < 3$. 3+3+4

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