B.SC. SECOND SEMESTER (HONOURS) EXAMINATIONS, 2021

Subject: Mathematics **Course ID: 22114**

Course Code: SH/MTH/203/GE-2 **Course Title: Real Analysis**

Full Marks: 40 **Time: 2 Hours**

The figures in the margin indicate full marks

Notations and symbols have their usual meaning

1. ***Answer any five questions:*** **2×5=10**

(a) Prove that the intersection of an infinite number of open sets in $\mathbb{R}$ need not be open in $\mathbb{R}$.

(b) Verify the Bolzano-Weierstrass theorem for the set $S = \left\{(-1)^n\left(1+\frac{1}{n}\right) : n \in \mathbb{N}\right\}$.

(c) Give examples of divergent sequences $\{u_n\}$ and $\{v_n\}$ such that the sequence $\{u_n + v_n\}$ is convergent.

(d) Examine if the set S is closed in $\mathbb{R}$, where $S = \{x \in \mathbb{R}: sin\, x = 0\}$.

(e) Give an example of an open cover of the set $(0, 5]$ which does not have a finite subcover.

(f) Find the upper and lower limits of the sequence $\{u_n\}$, where $u_n = \frac{(-1)^n}{n} + \sin\left(\frac{n\pi}{2}\right)$.

(g) If Σu_n is a convergent series of positive real numbers, then prove that $\Sigma {u^2}_n$ is also convergent.

(h) Discuss the convergence of the series $\Sigma \frac{1}{n \log n (log log\, n)}$, n>2.

2. ***Answer any four questions*** : **5×4=20**

(a) (i) If $x > 0$, show that there exists a natural number n such that $0 < \frac{1}{n} < x$.

(ii) If $S \subset \mathbb{R}$ and $T \subset \mathbb{R}$ be open and closed in $\mathbb{R}$ respectively, then prove that $S - T$ is an open set and $T - S$ is a closed set in $\mathbb{R}$. 2+3

(b) (i) Test the convergence of the series $a + b + a^2 + b^2 + a^3 + b^3 + \cdots$, where 0<a<b<1.

(ii) Let $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots.\right\}$. Find S'. Show that S is not closed in $\mathbb{R}$. 3+2

(c) (i) Show that the set of rational numbers is enumerable.

(ii) Prove that $\lim_{n\to\infty} \frac{4^{3n}}{3^{4n}} = 0$. 3+2

(d) (i) Use Sandwich theorem to prove that $\lim_{n\to\infty}\left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}\right] = 0$.

(ii) Show that the series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots.$ is conditionally convergent. 3+2

(e) (i) Show that a constant sequence is convergent.

(ii) Prove that the sequence $\{u_n\}$ satisfying the condition $|u_{n+2} - u_{n+1}| \leq \frac{1}{2}|u_{n+1} - u_n|$ for all $n \in \mathbb{N}$ is a Cauchy sequence. 1+4

(f) If the sequences $\{u_n\}$ and $\{v_n\}$ converge to l and m respectively, then show that $\lim_{n\to\infty} \frac{1}{n}(a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1) = lm$.

3. *Answer any one question :* **10×1=10**

(a) (i) Prove that the sequence $\{u_n\}$ defined by $u_1 = \sqrt{6}$ and $u_{n+1} = \sqrt{6 + u_n}$ for all $n \geq 1$ converges to *3*.

(ii) Find the sup$\{u_n\}$ and inf$\{u_n\}$, where $u_n = \frac{(-1)^n}{n} + \sin\frac{n\pi}{2}$.

(iii) Let S be a non empty subset of $\mathbb{R}$, bounded below and $T = \{-x : x \in S\}$. Prove that the set T is bounded above and sup$T = -$inf$S$. 5+2+3

(b) (i) Let $\{u_n\}$ and $\{v_n\}$ be two bounded real sequences and $u_n > 0, v_n > 0$ for all $n \epsilon N$. Then prove that $\limsup u_n . \limsup v_n \geq \limsup u_n v_n$.

(ii) Prove that the intersection of an arbitrary collection of closed sets in $\mathbb{R}$ not necessarily a closed set in $\mathbb{R}$.

(iii) Examine the convergence of the series $1 + \frac{2^2}{3^2} + \frac{2^2}{3^2}\frac{4^2}{5^2} + \frac{2^2}{3^2}\frac{4^2}{5^2}\frac{6^2}{7^2} + \cdots$. 4+3+3

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