B.SC. SECOND SEMESTER (HONOURS) EXAMINATIONS, 2021

Subject: MATHEMATICS **Course ID: 22112**

Course Code: SH/MTH/202/C-4 **Course Title: Differential Equations & Vector Calculus**

Full Marks: 40 **Time: 2 Hours**

The figures in the margin indicate full marks

Unless otherwise mentioned the symbols have their usual meaning

1. Answer any five (05) questions. $2 \times 5 = 10$

a) Does the initial value problem (IVP):

$$\frac{dy}{dx} = 2x\sqrt{y}, \quad y(0) = 0$$

have a unique solution? Justify your answer.

b) Show that $y = x$ and $y = x^2$ are two linearly independent solutions of the differential equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0, \qquad (x \neq 0).$$

c) Evaluate: $\int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right)dt$, where $\vec{r} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$.

d) Find out the equilibrium points of the non-linear differential equation:

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + (3x^3 + x^2 - 2x) = 0.$$

e) Find the value of μ so that the vectors $(2\hat{\imath} - \hat{\jmath} + \hat{k})$, $(\hat{\imath} + \mu\hat{\jmath} - 3\hat{k})$ and $(3\hat{\imath} - 4\hat{\jmath} + 5\hat{k})$ are coplanar.

f) Determine whether the following functions are linearly dependent or independent: $f(x) = e^x$, $g(x) = e^{-2x}$, $h(x) = xe^{-2x}$.

g) Test the continuity of the vector function $\vec{f}(t)$ given by

$$\vec{f}(t) = |t|\hat{\imath} - \sin t\,\hat{\jmath} + (1 + \cos t)k \text{ at } t = 0.$$

h) Evaluate:

$$\int_1^2 \left(\vec{f} \cdot \frac{d\vec{f}}{dt}\right) dt,$$

where $\vec{f}(1) = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$ and $\vec{f}(2) = 2\hat{\imath} - 3\hat{\jmath} + \hat{k}$.

2. Answer any four (04) questions. **$5 \times 4 = 20$**

a) i) State the principle of superposition of solutions of a second order linear homogeneous differential equation. 1

ii) Show that the Wronskian $W(y_1, y_2)$ of any two solutions y_1 and y_2 of the differential equation

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = 0$$

on an interval I is $W = C_0 e^{-\int a(x)dx}$, where C_0 is a constant.

Also show that the Wronskian W is either identically zero on I or else is never zero on I. 3+1

b) Solve the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{e^x}{1+e^x}$$

by method of variation of parameters.

c) (i) If $\frac{d\vec{a}}{dt} = \vec{r} \times \vec{a}$ and $\frac{d\vec{b}}{dt} = \vec{r} \times \vec{b}$, then show that $\frac{d}{dt}(\vec{a} \times \vec{b}) = \vec{r} \times (\vec{a} \times \vec{b})$, where $\vec{r}$ is a constant vector and $\vec{a}$, $\vec{b}$ are vector functions of a scalar variable t.

(ii) If $\vec{\alpha} \neq \vec{0}$ but $\vec{\alpha}.\vec{\beta} = \vec{\alpha}.\vec{\gamma}$ and $\vec{\alpha} \times \vec{\beta} = \vec{\alpha} \times \vec{\gamma}$ then prove that $\vec{\beta} = \vec{\gamma}$. 3+2

d) Solve the differential equation $(x^2D^2 - xD + 3)y = x^2 \log x$, where $D \equiv \frac{d}{dx}$.

e) Solve the equation $(D^2 - 4D + 4)y = (x + x^3)e^{2x}$ $\left(where\ D \equiv \frac{d}{dx}\right)$ by method of undetermined coefficients.

f) Find the characteristic equation associated with the following linear homogeneous system of differential equations

$$\frac{dx}{dt} = 5x - 2y$$
$$\frac{dy}{dt} = 4x - y$$

also find the eigenvalues and corresponding eigen vectors.

Hence obtain the general solution of the above system of differential equations. 2+2+1

3. Answer any one (01) question. **$10 \times 1 = 10$**

a) i) Solve by method of variation of parameters $x^3\frac{d^3y}{dx^3} + x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x\log x \quad (x > 0)$

it being given that $y = x^{-1}, y = x, y = x^2$ are three linearly independent solutions of its reduced equation. 6

ii) Show that the necessary and sufficient condition that a proper vector $\vec{A}(t)$ to have a constant direction is $\vec{A} \times \frac{d\vec{A}}{dt} = 0.$ 4

b) i) Show that $x = 0$ is an ordinary point of the differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - y = 0.$$

Hence find the power series solution of the above equation about $x = 0$. 1+6

ii) If $\vec{r} = (a\cos t)\hat{i} + (a\sin t)\hat{j} + (at^2 \sec\alpha)\hat{k}$, then find the value of

$\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|$ and $\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right]$, where a and α are constants. 3

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