B.SC. SECOND SEMESTER (HONOURS) EXAMINATIONS, 2021

Subject: Mathematics **Course ID: 22111**

Course Code: SH/MTH/201/C-3 **Course Title: Real Analysis**

Full Marks: 40 **Time: 2 Hours**

The figures in the margin indicate full marks

Notations and symbols have their usual meaning

1. **Answer *any five* of the following questions:** **(2 x 5 = 10)**

a) Explain why the Bolzano Weierstrass theorem (for sets as well as for sequences) does not hold in the system of rational numbers.

b) If no points of a set $S \subseteq \mathbb{R}$ are its limit points then prove that S is at most countable.

c) Let $S \subseteq \mathbb{R}$ and $S + a = \{x + a, x \in S, a \in \mathbb{R}\}$. Show that $\sup(S + a) = \sup S + a$.

d) Prove that if x is an arbitrary real number, there is a sequence $\{r_n\}_n$ of rational numbers converging to x.

e) Give an example with justification of a set in $\mathbb{R}$ which is neither open nor closed.

f) Show that the sequence $\{ S_n \}$ is convergent where $S_n = \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$.

g) Check whether the sequence $\{S_n\}$ where $S_{n+1} = \sqrt{3S_n}$ and $S_1 = 1$ is bounded and monotonic.

h) Test the convergence of the series $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \cdots$

2. **Answer *any four* of the following questions:** **(5 x 4= 20)**

a) Using Cauchy's integral test discuss the convergence of the following series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$

for different values of p.

b) Let A be a nonempty subset of $\mathbb{R}$ and x be any real number. Let us define $d(x, A) = inf\,\{|x - a| : a \,\epsilon A\}$. Show that $d(x, A) = 0$ if and only if $\epsilon \bar{A}$, where $\bar{A}$ denotes the closure of A.

c) Discuss the convergence or divergence of $\{x_n\}_n$ where

$$x_n = \frac{\alpha^n - \beta^n}{\alpha^n + \beta^n}$$

where α and β are real numbers such that $|\alpha| \neq |\beta|$.

d) Let $S_1 = \{x \in \mathbb{R} : \sin x = 0\}$ and $S_2 = \left\{x \in \mathbb{R} : \cos\frac{1}{x} = 0\right\}$. Test whether the sets S_1 and S_2 are open or closed.

e) If $\{P_n\}_n$ is a monotone non-increasing sequence of positive terms such that $\lim_{n\to\infty} P_n = 0$, show that the series

$$P_1 - \frac{1}{2}(P_1 + P_3) + \frac{1}{3}(P_1 + P_3 + P_5) + \cdots$$

is convergent.

f) If $x \in \mathbb{R}^+$ then show that there exists $a \in \mathbb{R}$ such that $a^2 = x$ where $\mathbb{R}^+$ is the set of all positive real numbers.

3. Answer *any one* of the following questions: (10 x 1 = 10)

a) (i) Without assuming any theorem or result, prove that every closed subset of a compact set is a compact set.

(ii) Suppose that $\{S_n\}_n$ is a sequence for which there exists a constant *c* such that

$$\forall\, n \in \mathbb{N}, \qquad |S_{n+1} - S_n| < \frac{c}{2^n}.$$

Show that $\{S_n\}_n$ is a Cauchy sequence. 5+5

b) (i) Prove that a bounded sequence is convergent if and only if $\overline{lim} a_n = \underline{lim} a_n$.

(ii) Prove that every finite subset of $\mathbb{R}$ is a closed set and it has no limit point.

(iii) Using Cauchy's general principle, prove that the sequence $\{ a_n\}$ is not convergent where $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. 3+3+4

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